

УДК 512.53

## Bidiagonal Ranks of Completely (0-)simple Semigroups

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Received 12.09.2015, received in revised form 19.01.2016, accepted 24.02.2016

A *bidiagonal act* over a semigroup is a two-sided act, where the semigroup acts on its Cartesian power. A *bidiagonal rank* of a semigroup is the least power of a generating set of the bidiagonal act over this semigroup. In this paper we compute bidiagonal ranks of completely (0-)simple semigroups.

**Keywords:** act over a semigroup, diagonal rank, completely (0-)simple semigroup.

DOI: 10.17516/1997-1397-2016-9-2-144-148.

A *right act* over a semigroup  $S$  is a set  $X$  with a map  $X \times S \rightarrow X$ ,  $(x, s) \mapsto xs$  satisfying  $(xs)s' = x(ss')$  for all  $x \in X$ ,  $s, s' \in S$  (see [1]). A *left  $S$ -act*  $Y$  over the semigroup  $S$  is defined analogously:  $S \times Y \rightarrow Y$ ,  $(s, y) \mapsto sy$ ,  $s(s'y) = (ss')y$  for all  $s, s' \in S$ ,  $y \in Y$ . Let  $S, T$  be semigroups. A set  $Z$  is called an  $(S, T)$ -*act* (*bi-act* over  $S$  and  $T$ ), if it is a left  $S$ -act and a right  $T$ -act at the same time and  $(sz)t = s(zt)$  for all  $z \in Z$ ,  $s \in S$ ,  $t \in T$ . The right  $S$ -act  $X$ , left  $S$ -act  $Y$  and  $(S, T)$ -bi-act  $Z$  may be denoted by  $X_S$ ,  ${}_SY$ , and  ${}_SZ_T$ .

A generating set  $G$  of the act  $(S \times S)_S$  is called *irreducible* if none of its subsets  $G' \subset G$  is a generating set of this act. Clearly, any finite generating set may be reduced to an irreducible one. A generating set is called *minimal* if it is minimal with respect to power.

Note that a diagonal act over a semigroup is a unary algebra. Indeed, if  $S$  is a semigroup, then multiplication by  $s \in S$  may be thought as applying unary operation  $\varphi_s : x \mapsto xs$ , where  $x \in S$ . Therefore, the following theorem is applicable to diagonal acts.

**Theorem 1** (Kartashov, [2], Theorem 1). *Let  $A$  be an algebra with signature  $\Sigma = \{\varphi_i \mid i \in I\}$ , where all operations  $\varphi_i$  are unary. If  $A$  is finitely generated, then any irreducible generating set of  $A$  is minimal.*

Let  $S$  be a semigroup. A *right diagonal rank* of  $S$  (denoted by  $\text{rdr } S$ ) is called the least power of generating sets of the diagonal right act of  $S$ , or

$$\text{rdr } S = \min \{|A| \mid A \subseteq S \times S \wedge AS^1 = S \times S\}.$$

A *bidiagonal rank*  $\text{bdr } S$  of  $S$  is defined in a similar way.

Diagonal acts were used in [3, 4] to study conditions of wreath products to be finitely generated. Diagonal acts themselves became a subject of study in [5–7] and others. In these papers the prime problem were conditions of finite generateness of infinite diagonal acts. The notion of diagonal rank was explicitly formulated in the paper [8]. In the paper [9] one-sided diagonal ranks of completely (0-)simple semigroups were calculated. In this paper we continue the study of diagonal ranks of semigroups and calculate bidiagonal ranks of completely (0-)simple semigroups. For completeness we cite the prime results of [9] concerned with completely (0-)simple

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semigroups. These theorems may be proven using reasoning similar to the one in the presented theorems.

**Theorem 2** ([9], Theorem 2). *Let  $S$  be a Rees matrix semigroup with sandwich-matrix  $P$ :  $S = \mathcal{M}(G, I, \Lambda, P)$ . Then the right diagonal rank of  $S$  equals  $|I|^2|G|$ , if  $\Lambda$  is singleton and  $|I|^2|G|^2\Lambda(\Lambda - 1)$  otherwise.*

**Theorem 3** ([9], Theorem 3). *Let  $S$  be a Rees matrix semigroup with zero:  $S = \mathcal{M}^0(G, I, \Lambda, P)$ . Let  $|I| = k$ ,  $|G| = t$ ,  $|\Lambda| = l$ . If  $l \geq 2$  then:*

- if there are no zeros in  $P$ , then  $\text{rdr } S = k^2t^2(l^2 - l) + 2k$ ;
- if there are zeros in  $P$ , but there is no column with two or more nonzero elements, then  $\text{rdr } S = k^2t^2(l^2 - l) + k^2t$ ;
- otherwise,  $\text{rdr } S = k^2t^2(l^2 - l)$ .

If  $l = 1$ , then  $\text{rdr } S = k^2t + 2k$ .

The following theorems are the body of this paper. As in the case of one-sided ranks, bidiagonal ranks of completely (0-)simple semigroups depend on sandwich-matrices insignificantly.

**Theorem 4.** *Let  $S$  be a completely simple semigroup:  $S = \mathcal{M}(G, I, \Lambda, P)$ , where  $G$  is a group of  $t$  elements and  $f$  conjugacy classes,  $I$  and  $\Lambda$  are index sets of  $k$  and  $l$  elements correspondingly and  $P$  is a sandwich-matrix. Then*

- if  $k, l \neq 1$ , then  $\text{bdr } S = t^2l(l - 1)k(k - 1)$ ;
- if  $k = 1, l \neq 1$ , then  $\text{bdr } S = tl(l - 1)$ ;
- if  $l = 1, k \neq 1$ , then  $\text{bdr } S = tk(k - 1)$ ;
- if  $l = 1, k = 1$ , then  $\text{bdr } S = f$ .

*Proof.* Let  $k, l > 1$ ,  $M = \{((x)_{i_1\lambda_1}, (y)_{i_2\lambda_2}) \mid i_1 \neq i_2, \lambda_1 \neq \lambda_2\}$ . We prove that  $M$  is an irreducible generating set of  ${}_S(S \times S)_S$ . Indeed, from

$$(a)_{i\lambda} ((x)_{i_1\lambda_1}, (y)_{i_2\lambda_2}) = ((ap_{\lambda i_1}x)_{i\lambda_1}, (ap_{\lambda i_2}y)_{i\lambda_2}),$$

and

$$((x)_{i_1\lambda_1}, (y)_{i_2\lambda_2})(b)_{i\lambda} = ((xp_{\lambda i_1}b)_{i_1\lambda}, (yp_{\lambda i_2}b)_{i_2\lambda}),$$

we see that pairs from  $M$  cannot be obtained from any other pairs using one-sided or two-sided multiplication. Therefore,  $M$  is a subset of any generating set.

Now we prove that  $M$  itself is a generating set. A pair of a kind  $((u)_{j_1\mu}, (v)_{j_2\mu})$ , where  $j_1 \neq j_2$ , may be obtained from  $M$  in the following way. Take  $\lambda_1 \neq \lambda_2 \in \Lambda$  and any  $i \in I$ . Then

$$\left((1)_{j_1\lambda_1}, (vu^{-1}p_{\lambda_1 i}p_{\lambda_2 i}^{-1})_{j_2\lambda_2}\right)(p_{\lambda_1 i}^{-1}u)_{i\mu} = ((u)_{j_1\mu}, (v)_{j_2\mu}).$$

In a similar way we may get pairs of a kind  $((u)_{i\lambda_1}, (v)_{i\lambda_2})$ , where  $\lambda_1 \neq \lambda_2$ .

Pairs of a kind  $((u)_{j\mu}, (v)_{j\mu})$  are obtained by multiplying the pair

$$\left((p_{\lambda_1 i_2}^{-1}up_{\lambda_2 i_4}^{-1})_{i_2\lambda_2}, (p_{\lambda_1 i_3}^{-1}vp_{\lambda_3 i_4}^{-1})_{i_3\lambda_3}\right)$$

by elements  $(1)_{j\lambda_1}$  and  $(1)_{i_4\mu}$ . Here  $i_2 \neq i_3$ , and  $\lambda_2 \neq \lambda_3$ . These pairs are in  $M$  and  $M$  is a generating set. Hence,  $\text{bdr } S = |M| = t^2l(l - 1)k(k - 1)$ .

Let  $l = 1, k > 1$ . Matrix  $P$  is a vector. Using Lemma 3.6 from [10] and the remark after we can say that  $P$  consists of 1's. Let  $M = \left\{ \left( (1)_i, (t)_j \right) \mid i \neq j \right\}$ .

Take a pair  $x = \left( (a)_i, (b)_j \right)$  such that  $i \neq j$ . Since

$$\left( (1)_i, (ba^{-1})_j \right) (a)_i = \left( (a)_i, (b)_j \right),$$

then  $x \in MS$ . Pairs  $((a)_i, (b)_i)$  we get via

$$(1)_i \left( (1)_i, (ba^{-1})_j \right) (a)_i = ((a)_i, (b)_i).$$

Hence  $S \times S \subseteq S^1 MS$  and  $M$  is a generating set. Now we prove that  $M$  is irreducible. It is sufficient to show that no pairs from  $M$  are products of an another pair and elements of  $S^1$ . Let  $\left( (1)_i, (a)_j \right) = s' \left( (1)_p, (b)_q \right) s$  for some  $s, s' \in S^1$ , where  $i \neq j, p \neq q$ . Let  $s = (c)_r$ . Then  $\left( (1)_i, (a)_j \right) = \left( (1)_p, (b)_q \right) (c)_r = \left( (c)_p, (bc)_q \right)$ . Hence  $p = i, q = j, c = 1$  and  $\left( (1)_p, (b)_q \right) = \left( (1)_i, (a)_j \right)$ . It means that  $M$  is irreducible, so  $\text{bdr } S = |M| = tk(k-1)$ .

Let  $k = 1, l \neq 1$ . In a similar way one can prove that  $\text{bdr } S = tl(l-1)$ .

Let  $k = 1, l = 1$ . Now  $S \simeq G$ . Using Lemma 4.1 from [5] we get  $\text{bdr } G = f$ . So  $\text{bdr } S = f$ .  $\square$

**Theorem 5.** Let  $S$  be a completely (0-)simple semigroup:  $S = \mathcal{M}^0(G, I, \Lambda, P)$ , where the group  $G$  consists of  $t$  elements and  $f$  conjugacy classes,  $I$  and  $\Lambda$  are index sets of  $k$  and  $l$  and elements correspondingly and  $P$  is a sandwich matrix

- Let  $k, l > 1$ . If a row or a column of  $P$  has two or more non-zero elements we say that it is good. Consider the following cases.
  1. There are no zeros in  $P$ . Then  $\text{bdr } P = t^2 k(k-1)l(l-1) + 2$ .
  2. There are zeros in  $P$ . Moreover, there is a good row and a good column in  $P$ . Then  $\text{bdr } P = t^2 k(k-1)l(l-1)$ .
  3. There is a good row in  $P$ , but no good column. Then  $\text{bdr } P = t^2 k(k-1)l(l-1) + tk(k-1)$ .
  4. There is a good column in  $P$ , but no good row. Then  $\text{bdr } P = t^2 k(k-1)l(l-1) + tl(l-1)$ .
  5. There are no good rows or columns in  $P$ . Then  $\text{bdr } P = t^2 k(k-1)l(l-1) + tk(k-1) + tl(l-1) + f$ .
- If  $k = 1, l > 1$ , then  $\text{bdr } S = tl(l-1) + 2$ .
- If  $l = 1, k > 1$  then  $\text{bdr } S = tk(k-1) + 2$ .
- If  $k = l = 1$ , then  $\text{bdr } S = f + 2$ .

*Proof.* Throughout the following text consider the indices denoted by different symbols to be different.

Let  $k, l > 1$ . Action in the biact  ${}_S(S \times S)_S$  is defined by the following expressions:

$$\begin{aligned} (a)_{i\lambda} \left( (x)_{i_1\lambda_1}, (y)_{i_2\lambda_2} \right) &= (ap_{\lambda i_1} x)_{i\lambda_1}, (ap_{\lambda i_2} y)_{i\lambda_2}, \\ \left( (x)_{i_1\lambda_1}, (y)_{i_2\lambda_2} \right) (b)_{i\lambda} &= (xp_{\lambda_1 i} b)_{i_1\lambda}, (yp_{\lambda_2 i} b)_{i_2\lambda}. \end{aligned}$$

Divide all pairs from  $S \times S$  into the following classes:

1.  $\left( (u)_{j_1\mu_1}, (v)_{j_2\mu_2} \right)$ ;    2.  $\left( (u)_{j\mu_1}, (v)_{j\mu_2} \right)$ ;    3.  $\left( (u)_{j_1\mu}, (v)_{j_2\mu} \right)$ ;
4.  $\left( (u)_{j\mu}, (v)_{j\mu} \right)$ ;    5.  $\left( (u)_{j\mu}, 0 \right)$ ;    6.  $\left( 0, (v)_{j\mu} \right)$ ;    7.  $(0, 0)$ .

Pairs of the class 1 do not belong to  $A = S^1 \cdot (S \times S) \cdot S^1$ . So we add them to the generating set. There are  $t^2k(k-1)l(l-1)$  such pairs.

Consider the class 2. Depending on  $P$  we have two cases.

- There is a good row in  $P$ . Then pairs of the class 2 are obtainable via left multiplication of the class 1 pairs.
- There are no good rows in  $P$ . Then the class 2 pairs are not obtainable from the class 1 pairs. To fix that we add to the generating set the pairs of a kind  $\left((1)_{q\lambda_1}, (y)_{q\lambda_2}\right)$ , where index  $q$  is arbitrary, but fixed. There are  $tl(l-1)$  such pairs.

Consider pairs of the class 3. Depending of  $P$  we have the following cases.

- There is a good column in  $P$ . Pairs of the class 3 are obtainable from the class 1 pairs via right multiplication.
- There is no good columns in  $P$ . Then the class 2 pairs are not obtainable from the class 1 pairs. To fix that we add to the generating set the pairs of a kind  $\left((1)_{i_1\mu}, (y)_{i_2\mu}\right)$ , where index  $\mu$  is arbitrary, but fixed. There are  $tk(k-1)$  such pairs.

Consider pairs of the class 4. Depending on  $P$  we have the following cases.

- There is a good row and a good column in  $P$ . Then pairs of the class 4 are obtainable via two-sided action on the class 1 pairs.
- There is no good row in  $P$ , but there is a good column. Then the pairs of the class 4 are obtainable via right-sided action on the class 2 pairs. These pairs we obtain from the generating set.
- There is no good column in  $P$ , but there is a good row. Then the pairs of the class 4 are obtainable via the left-sided action on the class 3 pairs. These pairs we obtain from the generating set.
- There are no good rows or columns in  $P$ . Fix indices  $q, \mu$ , pick one representative  $g_r$  from every conjugacy class of  $G$  and add to the generating set  $f$  pairs  $\left((1)_{q\mu}, (g_r)_{q\mu}\right)$ ,  $1 \leq r \leq f$ .

We show how to get an arbitrary pair  $\left((u)_{j\nu}, (v)_{j\nu}\right)$  from these pairs. Choose  $h, a, b \in G$  such that  $h^{-1}g_rh = vu^{-1}$ ,  $a = h^{-1}p_{\lambda q}^{-1}$ ,  $b = p_{\mu i}^{-1}hu$ . Then

$$(a)_{j\lambda} \left( (1)_{q\mu}, (g_r)_{q\mu} \right) (b)_{i\nu} = \left( (u)_{j\nu}, (v)_{j\nu} \right).$$

Consider class 5 and class 6 pairs. If there are no zeroes in  $P$  we have to add two pairs to the generating set:  $\left((x)_{j\mu}, 0\right)$  и  $\left(0, (x)_{j\mu}\right)$ .

The pairs  $(0, 0)$  are obtainable from any pair.

So we have the following cases.

1. There are no zeroes in  $P$ . Then  $\text{bdr } P = t^2k(k-1)l(l-1) + 2$ .
2. There is a good column and a good row in  $P$ . Then  $\text{bdr } P = t^2k(k-1)l(l-1)$ .
3. There is a good row in  $P$ , but no good column. Then  $\text{bdr } P = t^2k(k-1)l(l-1) + tk(k-1)$ .
4. There is a good column in  $P$ , but no good row. Then  $\text{bdr } P = t^2k(k-1)l(l-1) + tl(l-1)$ .
5. There are no good rows or columns in  $P$ . Then  $\text{bdr } P = t^2k(k-1)l(l-1) + tk(k-1) + tl(l-1) + f$ .

Let  $l = 1$ ,  $k > 1$ . Then  $S$  is a left group with externally adjoined zero:  $S = A \cup \{0\} = \{(g)_i \mid g \in G, i \in I\} \cup \{0\}$ , where  $|A| = k > 1$ . By Theorem 4 the act  ${}_A(A \times A)_A$  is generated by the pairs  $((1)_i, (g)_j)$ , where  $g \in G$ ,  $i, j \in I$ . To get an irreducible generating set of the act  ${}_S(S \times S)_S$  we add the pairs  $(0, (1)_{i_0})$  and  $((1)_{i_0}, 0)$ . Hence  $\text{bdr } S = tk(k-1) + 2$ .

Analogously for the case  $k = 1$ ,  $l > 1$  we get  $\text{bdr } S = tl(l-1) + 2$ .

Let  $k = l = 1$ . Now we need only  $f$  pairs  $(1, g_r)$ ,  $1 \leq r \leq f$  to get pairs  $(u, v)$  and two pairs  $(1, 0)$  and  $(0, 1)$  to get pairs  $(u, 0)$  and  $(0, u)$ .  $\square$

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## Бидиагональные ранги вполне (0-)простых полугрупп

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*Диагональным биполигоном над полугруппой называется полигон, в котором полугруппа действует с двух сторон на свою декартову степень. Бидиагональным рангом полугруппы называется наименьшая мощность порождающего множества ее бидиагонального полигона. В данной работе мы вычисляем бидиагональные ранги вполне (0-)простых полугрупп.*

*Ключевые слова:* полигон над полугруппой, диагональный ранг, вполне (0-)простая полугруппа.